stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 132/79 NOVEMBER

W.G. VALIANT

ON AN IDENTITY OF T. BANG

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

1980 Mathematics subject classification: 15A15, 05B20

On an identity of T. Bang

by

W.G. Valiant

ABSTRACT

Let $\phi_{n,k}$ be the class of n×n matrices with nonnegative integer coefficients such that each row and column sum equals k. By the "tensor product identity" of T. Bang we prove that

$$\min\{\text{per } \alpha \mid \alpha \in \phi_{n,4}\} \geq 2.(\tfrac{3}{2})^n$$

$$\min\{\text{per } \alpha \mid \alpha \in \phi_{n,6}\} \geq 5.(\tfrac{20}{9})^{n-1}.$$

KEY WORDS & PHRASES: *permanents*

1. INTRODUCTION

Let $\phi_{n,k}$ be the class of $n\times n$ matrices α with nonnegative integer coefficients such that each row and column sum of α equals k. We denote the permanent of α by per α and define

$$\lambda_k(n) := \min_{\alpha\in\phi_{n,k}} \text{per}\ \alpha; \qquad \theta_k := \inf_n (\lambda_k(n))^{1/n}.$$

Trivially, $\lambda_1(n) = 1$ and $\lambda_2(n) = 2$, so $\theta_1 = \theta_2 = 1$. ERDÖS and RÉNYI [2] conjectured that $\theta_k > 1$ for $k \geq 3$. This was proved by M. Voorhoeve and T. Bang and S. Friedland. VOORHOEVE [5] showed that $\theta_3 \geq 4/3$ by elementary methods, whereas BANG [1] and FRIEDLAND [3] proved that the permanent of a doubly stochastic matrix is at least e^{-n}, which implies that $\theta_k \geq k/e$. The basic tool in Bang's and Friedland's papers is an identity due to Bang.

In this paper we use Bang's identity to derive the estimates

$$\theta_4 \geq 3/2; \qquad \theta_6 \geq 20/9.$$

These bounds are slightly better than 4/e and 6/e respectively. SCHRIJVER and VALIANT [4] proved

$$\theta_k \leq (k-1)^{k-1}/k^{k-2}, \tag{1}$$

showing, by VOORHOEVE [5], that $\theta_3 = 4/3$. We conjecture that (1) gives the correct value for all θ_k's.

2. THE BASIC IDENTITY

THEOREM 1. (T. Bang). *Let* J_k *be the* $k\times k$ *matrix with all entries* $1/k$. *Then for any* $n\times n$ *matrix* $A = (a_{ij})$, *the permanent of the tensor product* $A \otimes J_k$ *satisfies*

$$\text{per}(A \otimes J_k) = \frac{(k!)^{2n}}{k^{kn}} \sum_{n,k}^{(\alpha)} \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!),$$

where the sum $\Sigma_{n,k}^{(\alpha)}$ *ranges over all matrices* $\alpha = (\alpha_{ij})_{i,j=1}^{n}$ *in the class* $\phi_{n,k}$.

PROOF. See FRIEDLAND [3], Theorem 2.1, or rather BANG [1].

COROLLARY 1. *Using the same notation, we have*

$$\operatorname{per}(A \otimes J_{k-1})\operatorname{per}(A) = \frac{((k-1)!)^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k}^{(\alpha)} \operatorname{per}\alpha \cdot \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!).$$

PROOF. We use throughout the notation $\Sigma_{n,k}^{(\alpha)}$ for sums ranging over all matrices α in $\phi_{n,k}$. Moreover, any product Πg_{ij} ranges over all possible g_{ij}'s. By Theorem 1 and the definition of the permanent,

$$\operatorname{per}(A \otimes J_{k-1})\operatorname{per}(A) = \frac{((k-1)!)^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k-1}^{(\beta)} \Pi(a_{ij}^{\beta_{ij}}/\beta_{ij}!) \cdot \sum_{n,1}^{(\gamma)} a_{ij}^{\gamma_{ij}} =$$

$$= \frac{(k-1)!^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k}^{(\alpha)} q(\alpha) \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!),$$

where

$$q(\alpha) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta\in\phi_{n,k-1},\gamma\in\phi_{n,1}}} \Pi(\alpha_{ij}!/\beta_{ij}!).$$

Clearly, $q(\alpha) = \operatorname{per}\alpha$. $\square$

3. DERIVATION OF THE ESTIMATES FOR θ_4 AND θ_6

THEOREM 2. $\lambda_4(m) \geq 2.(\frac{3}{2})^m$.

COROLLARY 2. $\theta_4 \geq 3/2$.

PROOF. Take k = 2 in Theorem 1 and Corollary 1. Since $\lambda_2(n) = 2$, we find

$$(2) \qquad \operatorname{per}(A \otimes J_2) << \frac{1}{2}(\operatorname{per} A)^2.$$

Here the sign << means term-by-term inequality: if we look upon per $A \otimes J_2$ and $\frac{1}{2}(\operatorname{per} A)^2$ as polynomials in the variables a_{ij}, each coefficient of the former is majorated by the latter.

Now take $A = B \otimes J_2$ for some m×m matrix $B = (b_{ij})$. If we consider both sides of (2) as polynomials in the b_{ij}'s, the same term-by-term inequality remains valid, hence

$$\text{(3)} \qquad \operatorname{per}(B \otimes J_4) = \operatorname{per}((B \otimes J_2) \otimes J_2) \ll \frac{1}{2}(\operatorname{per}(B \otimes J_2))^2$$

$$\ll \frac{1}{4} \operatorname{per}(B \otimes J_2)(\operatorname{per} B)^2.$$

By a reasoning similar to Corollary 1, we find

$$\operatorname{per}(B \otimes J_2)(\operatorname{per} B)^2 = \sum_{m,4}^{(\alpha)} r(\alpha) \Pi b_{ij}^{\alpha_{ij}}/\alpha_{ij}! \ ,$$

where

$$r(\alpha) = \sum_{\substack{\beta+\gamma+\delta=\alpha \\ \beta\in\Phi_{m,2},\gamma,\delta\in\Phi_{m,1}}} \Pi(\alpha_{ij}!/\beta_{ij}!) =$$

$$\text{(5)} \qquad = \sum_{\substack{\phi+\psi=\alpha \\ \phi\in\Phi_{m,3},\psi\in\Phi_{m,1}}} \operatorname{per} \phi.\Pi(\alpha_{ij}!/\phi_{ij}!) \le \max(\operatorname{per} \phi)\operatorname{per} \alpha$$

$$< (\operatorname{per} \alpha)^2.$$

By Theorem 1, the correct interpretation of the term-by-term inequality yields for each $\alpha \in \phi_{m,4}$

$$(\frac{9}{4})^m \le \frac{1}{4} r(\alpha) \le \frac{1}{4}(\operatorname{per} \alpha)^2. \qquad \Box$$

<u>THEOREM 3</u>. $\lambda_6(m) \ge 5.(\frac{20}{9})^{m-1}$.

<u>COROLLARY 3</u>. $\theta_6 \ge \frac{20}{9}$.

<u>PROOF</u>. By VOORHOEVE [5] we have for $n \ge 2$

$$\lambda_3(n) \ge 6.(\frac{4}{3})^{n-3} = \frac{81}{32} (\frac{4}{3})^n.$$

Hence, by Theorem 1 and Corollary 1, for $n \geq 2$

$$\operatorname{per}(A \otimes J_3) \ll \frac{32}{81} \operatorname{per}(A \otimes J_2)\operatorname{per}(A).$$

This implies for any $m\times m$ matrix B

$$\operatorname{per}(B \otimes J_6) \ll \frac{32}{81} \operatorname{per}(B \otimes J_4)\operatorname{per}(B \otimes J_2)$$

$$\ll \frac{16}{81} \operatorname{per}(B \otimes J_4)(\operatorname{per} B)^2.$$

As before,

$$\operatorname{per}(B \otimes J_4)(\operatorname{per} B)^2 \ll (\tfrac{9}{4})^m \cdot \sum_{m,6}^{(\alpha)} (\operatorname{per} \alpha)^2 \Pi(b_{ij}^{\alpha_{ij}}/\alpha_{ij}!)$$

So for all $\alpha \in \Phi_{m,6}$

$$(\tfrac{9}{4})^m (\operatorname{per} \alpha)^2 > \frac{81}{16} \cdot (\tfrac{100}{9})^m,$$

which proves the theorem. □

4. REMARKS

In (5) we used the estimate $\operatorname{per} \phi < \operatorname{per} \alpha$, a very crude approach, since $\phi \in \Phi_{n,3}$ and $\alpha \in \Phi_{n,4}$. It is not unreasonable to assume that if $\operatorname{per} \alpha = \lambda_4(m)$, $\operatorname{per} \phi$ is "about" $\lambda_3(m)$. This assumption then would imply that $\theta_4 = 27/16$. A similar heuristic reasoning holds for θ_6. This supports our conjecture.

<u>CONJECTURE 1</u>. $\theta_k = (k-1)^{k-1}/k^{k-2}$.

A stronger conjecture is

<u>CONJECTURE 2</u>. $\operatorname{per}(A \otimes J_k) \ll \operatorname{per}(A \otimes J_{k-1})\operatorname{per} A$.

<u>CONJECTURE 3</u>. For any $k\times k$ doubly stochastic matrix B and any $n\times n$ matrix A

$$\operatorname{per}(A \otimes B) \gg \operatorname{per}(A \otimes J_k)$$

Conjecture 3 implies both Conjecture 2 and the so-called van der Waerden conjecture, which is the case n = 1 of Conjecture 3.

REFERENCES

[1] BANG, T., *Matrixfunktioner som med et numerisk lille deficit viser v.d. Waerdens permanenthypothese,* to appear.

[2] ERDÖS, P. & A. RÉNYI, *On random matrices II,* Studia Sci. Math. Hungar. 3 (1968), 459-464.

[3] FRIEDLAND, S., *A lower bound for the permanent of a doubly stochastic matrix,* Annals of Math. 110, (1979), 167-176.

[4] SCHRIJVER, A. & W.G. VALIANT, *On lower bounds for permanents,* to appear.

[5] VOORHOEVE, M., *A lower bound for the permanents of certain* (0,1)-*matrices,* Proc. Kon. Ned. Akad. Wet., Ser. A 82 = Indag. Math. 41 (1979), 83-86.